JOURNAL OF APPROXIMATION THEORY 74, 41–58 (1993)

Monotonicity Preserving Subdivision Schemes

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Communicated by Allan Pinkus

Received May 24, 1991; accepted in revised form March 31, 1992

In this paper we discuss a class of subdivision schemes with a finite support suitable for curve design. We analyze the case where the masks of the scheme and the associated difference process are positive. We show that these schemes generate continuous functions of bounded variation, and that the monotonicity of the data is preserved. An estimate of the Lipschitz class of the generated functions is also obtained. For curves in R^d the control polygons generated by the scheme satisfy some variation diminishing properties, in particular, the arc-length is non-increasing. We characterize a particular subclass of schemes having bell-shaped refinable functions. Known sufficient conditions for excluding self-intersections and critical points of B-spline curves and surfaces hold also for these schemes.

1. INTRODUCTION

A binary subdivision scheme creates new control points from given ones by the rule

$$f_i^{n+1} = \sum_{j=-\infty}^{\infty} a_{i-2j} f_j^n, \qquad \{f_i^n\} \subset R^d, \quad \{a_i\} \subset R, \tag{1.1}$$

where the control points at level n, $\{f_i^n\}_{i=-\infty}^{\infty}$ are assigned to the binary mesh points $\{2^{-n}i\}_{i=-\infty}^{\infty}$.

Throughout this paper the mask $\{a_i\}$ is of compact support, and by the notation $\{a_i\}_{i=0}^{k}$ ($a_i = 0$ for $i<0$, $i>k$) we mean also $a_0 \neq 0$, $a_k \neq 0$. The number k is said to be the support of the scheme.

A detailed discussion of these schemes and a wide ranging list of references can be found in [1–5, 10].

A survey of the subdivision theory is found in [3], and the different contributors are mentioned there. Here, we will point to a reference only if the stated result is not found in [3], or does not follow immediately from the context there.

0021-9045/93 $5.00

In the following we present some notations and facts which are needed for the Introduction and then we state our results. Other preliminaries are found in Section 2.

Let f^n be the piecewise linear function defined by $f^n(2^{-n}i)=f_i^n$. A scheme is said to be C^v if for each set of initial values $\{f_i^0\}$ there exists $f\in C^v$ such that $\{f^n\}$ tends to f, the convergence is uniform on any finite interval, and $f\not\equiv 0$ at least for one choice of initial data.

Representing the data at level n by the generating function

$$F_n(z)=\sum_{j=-\infty}^{\infty} f_j^n z^j, \tag{1.2}$$

then the transformation (1.1) from level n to level $n+1$ is given by

$$F_{n+1}(z)=A(z)\,F_n(z^2), \tag{1.3}$$

where $A(z)$ is the characteristic polynomial of the scheme given by

$$A(z)=\sum_{j=0}^{k} a_j z^j. \tag{1.4}$$

Here and in the following, equalities of generating functions are defined by equalities in the coefficients of equal powers of z.

A necessary condition for a C^0 scheme is

$$\sum_i a_{2i}=\sum_i a_{2i+1}=1, \tag{1.5}$$

which is equivalent to

$$A(1)=2, \qquad A(-1)=0. \tag{1.6}$$

Backward differences at level n, $\Delta f_i^n=f_i^n-f_{i-1}^n$, have the generating function $(1-z)\,F_n(z)$, while backward divided differences are represented by $2^n(1-z)\,F_n(z)$. If the scheme satisfies (1.5) and if $k>1$, then the difference and the divided difference schemes exist, and have support $k-1$. The characteristic polynomial of the difference process is $(1+z)^{-1}A(z)$ and of the divided difference process is $2(z+1)^{-1}A(z)$.

From now on we denote the scheme by A, the divided difference process by dA, and the difference process by $\frac{1}{2}\,dA$.

Let A be C^0 then the limit curve produced by applying A to the initial data $\{f_i^0\}_{i=r_1}^{r_2}$ is given by

$$f(t)=\sum_{i=r_1}^{r_2} f_i^0 E(t-i), \qquad t\in[r_1+k-1,\, r_2+1], \tag{1.7}$$

where $E(t)$ satisfies the functional equation

$$E(t) = \sum_{i=0}^{k} a_i E(2t - i). \tag{1.8}$$

$E(t)$ is said to be the refinable function of A and is vanishing outside the interval $(0, k)$.

B-spline schemes constitute an example for the above description. A B-spline scheme of support k is defined by the mask

$$a_i = 2^{-(k-1)} \binom{k}{i}, \qquad 0 \leqslant i \leqslant k,$$

and its associated refinable function is the uniform normalized B-spline of order k with integer knots $\{0, ..., k\}$. The characteristic polynomial is

$$A(z) = 2^{-(k-1)} \sum_{i=0}^{k} \binom{k}{i} z^i = 2^{-(k-1)}(z+1)^k,$$

and the divided difference process is a B-spline scheme of support $k-1$. According to the convergence definition above the scheme is C^{k-2} (for $k=1$, the scheme does not converge). All results in this paper apply particularly to B-spline schemes.

In Section 3 we prove that if A has a positive mask ($a_i > 0$, $0 \leqslant i \leqslant k$) then the refinable function $E(t)$ is positive on $(0, k)$. This result was first conjectured in [10] and then proved by C. A. Micchelli and A. Pinkus in [9]. At the time of writing the paper, that proof was not known to us, however, our proof extends also to non-stationary schemes (see Remark 3.2).

The schemes we analyze in Sections 4 and 5 have a positive mask. In Section 4 we analyze the following class of subdivision schemes, which we denote by P^m (positive of order m).

DEFINITION 1.1. Let A be a subdivision scheme then $A \in P^m$, $m \geqslant 1$ if $d^{m-1}A$ satisfies (1.5) and the mask of $d^m A$ is positive.

Observe that if dA has a positive mask, then A also has a positive mask since multiplication by $(1+z)/2$ preserves the positivity. Thus we get

$$dA \in P^m \Leftrightarrow A \in P^{m+1}, \qquad m \geqslant 1, \tag{1.9}$$

and

$$P^{m+1} \subset P^m, \qquad m \geqslant 1. \tag{1.10}$$

EXAMPLE 1.2. Chaikin's scheme (the B-spline scheme of order 3) is given by

$$a_0 = \tfrac{1}{4} \qquad a_1 = \tfrac{3}{4} \qquad a_2 = \tfrac{3}{4} \qquad a_3 = \tfrac{1}{4}. \tag{1.11}$$

The divided difference process is given by

$$a_0 = \tfrac{1}{2} \qquad a_1 = 1 \qquad a_2 = \tfrac{1}{2}. \tag{1.12}$$

The second divided difference process is given by

$$a_0 = 1 \qquad a_1 = 1. \tag{1.13}$$

Schemes with support 1 have no difference process, thus Chaikin's scheme is P^2.

We show that P^m schemes are C^{m-1}. Moreover, the refinable function $E(t)$ satisfies $E^{(m-1)} \in BV$ and $E^{(m)} \in L^1$. The Lipschitz continuity of $E(t)$ is determined, and we get that $E \in \mathrm{LIP}_\gamma$ where

$$\gamma = m - 1 + \log_2(1/\|\tfrac{1}{2}\, d^m A\|_\infty). \tag{1.14}$$

Here $\|A\|_\infty$ means the ordinary sup norm of the linear operator A, given by (1.1). This estimate may be improved as we explain later.

P^m schemes are monotonicity preserving in the following sense. If the sequence $\{\Delta^{m-1} f\} \subset R$ is non-decreasing, then $f^{(m-1)}$ is non-decreasing. Here, and in the following, Δ^m denotes the backward difference operator of order m. We also show that P^m schemes are monotonicity preserving in a stricter sense.

P^m schemes satisfy some variation diminishing properties. Starting with the initial control polygon $\{f_i^0\}_{i=1}^r \subset R^d$ and applying a P^1 scheme, we show that

$$\sum_i \|\Delta f_i^{n+1}\| \leqslant \sum_i \|\Delta f_i^n\|, \tag{1.15}$$

where $\|\ \|$ is any semi-norm in R^d. In particular using the norm $\|\ \|_2$ we get that the arc-length is non-increasing, and using norm $\|\ \|_1$ with $d = 1$ then the total variation is non-increasing. Hence the arc-length of the limit curve and the total variation of its components are bounded by those of the initial data.

We conclude Section 4 by locating the zeros of the characteristic polynomial and by giving a different definition of P^m.

In Section 5 we discuss a class of schemes producing bell-shaped refinable functions according to the following definition:

DEFINITION 1.3. A function $E(t)$ of compact support $(0, k)$ is bell-shaped if

$$E(t) \in C^1 \tag{1.16}$$

$$E(t) > 0, \qquad t \in (0, k) \tag{1.17}$$

$$E\left(\frac{k}{2} + t\right) = E\left(\frac{k}{2} - t\right), \qquad t \in \left(0, \frac{k}{2}\right) \tag{1.18}$$

$$E'(t) > 0, \qquad t \in \left(0, \frac{k}{2}\right) \tag{1.19}$$

$$\sum_{i=-\infty}^{\infty} E(t-i) = 1. \tag{1.20}$$

In [7] it is shown that the analysis of self-intersections and critical points done in [6] for B-spline curves holds for curves of the form $\sum_{i=1}^{r} f_i^0 E(t-i)$, where $E(t)$ is bell-shaped.

Our main result in Section 5 states that if dA satisfies (1.5) and if the mask of dA is bell-shaped (in the sense of Definition 5.1) then $A \in P^2$ and the refinable function of A is bell-shaped.

2. PRELIMINARIES

In the following we present and clarify some well-known results about subdivision schemes (see also [3]).

Property 2.1. Let A be a scheme of support k then the limit values on an interval $[j2^{-n}, (j+1)2^{-n}]$, $j \in Z$ are determined by only k successive values at level n, namely, $\{f_i^n\}_{i=j-k+1}^{j}$. In particular, starting with a set of initial data $\{f_i^0\}_{i=r_1}^{r_2}$, then the interval of interest is $[r_1 + k - 1, r_2 + 1]$, since there the behaviour of the limit function is determined by $\{f_i^0\}_{i=r_1}^{r_2}$. The refinable function is the limit corresponding to $\{f_i^0\}_{i=-(k-1)}^{k-1} \subset R$, where $f_i^0 = \delta_{i,0}$.

Property 2.2. Each k successive values at level n determine $k+1$ successive values at level $n+1$, by the following relation:

$$\begin{bmatrix} f^{n+1}_{2j+k-1} \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ f^{n+1}_{2j+2k-2} \\ f^{n+1}_{2j+2k-1} \end{bmatrix} = \begin{bmatrix} a_{k-1} & a_{k-3} & \cdots & \cdots & \cdots & & 0 \\ a_k & a_{k-2} & \cdots & \cdots & \cdots & & \vdots \\ 0 & a_{k-1} & a_{k-3} & \cdots & \cdots & & \vdots \\ 0 & a_k & a_{k-2} & \cdots & \cdots & & \vdots \\ \vdots & & & & & & \vdots \\ \vdots & & & & & & 0 \\ \vdots & \cdots & \cdots & \cdots & a_3 & a_1 & 0 \\ \vdots & \cdots & \cdots & \cdots & \cdots & a_2 & a_0 \\ 0 & \cdots & \cdots & \cdots & \cdots & a_3 & a_1 \end{bmatrix} \begin{bmatrix} f^n_j \\ \cdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ f^n_{j+k-1} \end{bmatrix}. \tag{2.1}$$

The matrix (2.1) is given by

$$A_{ij} = a_{i-2j+k}, \qquad 1 \leqslant i \leqslant k+1, \quad 1 \leqslant j \leqslant k. \tag{2.2}$$

The following theorem is central for the smoothness analysis of schemes.

THEOREM 2.3. *Let dA be C^ν then A is $C^{\nu+1}$. Moreover, if A operating on $\{f_i^0\}$ converges to $f(t)$, then dA operating on $\{\Delta f_i^0 = f_i^0 - f_{i-1}^0\}$ converges to $f'(t)$.*

For continuity analysis we use the following result.

THEOREM 2.4. *Let A satisfy* (1.5); *then A is C^0 if and only if there exists $L \in Z^+$ and $0 < \alpha < 1$ such that*

$$\|(\tfrac{1}{2}\, dA)^L\|_\infty = \alpha. \tag{2.3}$$

Here $(\frac{1}{2}\, dA)^L$ means applying L times the linear operator $\frac{1}{2}\, dA$.
Moreover, the refinable function of A, $E(t)$, satisfies

$$E(t) \in \mathrm{LIP}_\gamma, \qquad \gamma = \log_2(1/\alpha^{1/L}). \tag{2.4}$$

Remark 2.5. The operator A^L is termed the L-iterated scheme of A, and is given by

$$f_i^{n+L} = \sum_{j=-\infty}^{\infty} a^{(L)}_{i-2^L j} f_j^n \tag{2.5}$$

$\{a_i^{(L)}\}_{i=0}^{N_L}$ is said to be the mask of A^L (N_L depends only on k and L).

Property 2.6. Let A be a scheme with a positive mask ($a_i > 0$, $0 \leqslant i \leqslant k$) satisfying (1.5). Then:

(i) A is a C^0 scheme (see [10]).

(ii) A satisfies the convex-hull property:

$$f(t)\in \mathrm{CONV}(\{f_i^n\}_{i=j+k-1}^{j}), \qquad t\in[j2^{-n},(j+1)2^{-n}]. \tag{2.6}$$

Here CONV denotes the set of all convex combinations.

(iii) The refinable function $E(t)$ satisfies

$$E(t)>0, \qquad 0<t<k \tag{2.7}$$

(see [9]).

An alternative proof of part (iii) will be given in Section 3.

3. Positivity of the Refinable Function

The following result was already proved in [9] and the proof here is an alternative one.

THEOREM 3.1. *Let the scheme A satisfy* (1.5) *and let the mask* $\{a_i\}_{i=0}^{k}$ *be positive then the refinable function of A, $E(t)$, is positive on* $(0,k)$.

Proof. $E(t)$ is the limit function corresponding to the initial data $\{f_i^0\}_{i=-(k-1)}^{k-1}\subset R$, $f_i^0=\delta_{i,0}$ and $E(t)$ is a continuous non-negative function on $(0,k)$ (see Properties 2.1, 2.6).

Applying one step of A then the data at level 1, $\{f_i^1\}_{i=-(k-1)}^{2(k-1)+1}$ is given by

$$\underbrace{0,\dots,0}_{k-1},\, a_0,\dots,a_k,\, \underbrace{0,\dots,0}_{k-1}. \tag{3.1}$$

(It is easy to see that the boundary layers in (3.1) are composed of $k-1$ zeros by applying the matrix A in (2.1) to the vectors $(1,0,\dots,0)$ and $(0,\dots,0,1)$.) Assume by induction that the data at level n, $\{f_i^n\}$, has the structure

$$\underbrace{\;0\;}_{Q_1^n}\underbrace{\;+\;}_{Q_2^n}\underbrace{\;0\;}_{Q_3^n}, \tag{3.2}$$

where the regions Q_1^n, Q_3^n contain $k-1$ zeros each. Q_2^n is composed of only positive numbers and has at least $k+1$ of them.

Let $k\geqslant 3$ be odd then each value at level $n+1$ is a positive linear combination of $(k+1)/2$ consecutive values at level n. By the induction hypothesis Q_1^n and Q_3^n do not take part in the same combination. Thus, the only zeros at level $n+1$ are produced by combinations of $Q_1^n(Q_3^n)$ alone, and since there are only $k-1$ such combinations of $Q_1^n(Q_3^n)$, then the induction is completed.

The proof of the induction for an even number k is analogous and the case $k=1$ is trivial.

The proof of the theorem is completed by the following argument. Fix $t \in (0, k)$, then there exists n such that $E(t)$ is determined by k consecutive values in Q_2^n, and by the convex hull property $E(t) > 0$. ∎

Remark 3.2. A scheme is said to be non-stationary if the transformation from level n to level $n+1$ depends on n. The proof here applies also for some cases of non-stationary schemes, since it does not assume that $E(t)$ satisfies a functional equation. For example consider the scheme

$$\begin{aligned} f_i^{n+1} &= \sum_{j=-\infty}^{\infty} a_{i-2j} f_j^n, \qquad n \text{ odd}, \\ f_i^{n+1} &= \sum_{j=-\infty}^{\infty} \tilde{a}_{i-2j} f_j^n, \qquad n \text{ even}, \end{aligned} \tag{3.3}$$

where $\{a_i\}_{i=0}^k$ and $\{\tilde{a}_i\}_{i=0}^k$ are positive masks. The limit corresponding to $f_i^0 = \delta_{i,0}$ is continuous by extending the arguments in [10]. Then using the above proof we deduce that $E(t) > 0$ on $(0, k)$.

4. Positivity of Order m

In the following we discuss properties of the difference process, $\frac{1}{2}\, dA$ where $A \in P^1$. The scheme $\frac{1}{2}\, dA$ is given by

$$\Delta f_i^{n+1} = \sum_{j \in Z} c_{i-2j} \Delta f_j^n, \tag{4.1}$$

where $\{c_i\}_{i=0}^{k-1}$ and the mask of A, $\{a_i\}_{i=0}^k$, are related by

$$a_i = c_i + c_{i-1}. \tag{4.2}$$

By summation on i we get

$$\sum_i a_i = 2 \sum_i c_i, \tag{4.3}$$

and in view of (1.5) we have

$$\sum_i c_i = 1. \tag{4.4}$$

In the following we estimate the operator norms $\|\frac{1}{2}\, dA\|_1$ and $\|\frac{1}{2}\, dA\|_\infty$.

THEOREM 4.1. *Let $A \in P^1$ then*

$$\|\tfrac{1}{2}\, dA\|_\infty = \max\left(\sum c_{2i}, \sum c_{2i+1}\right) < 1 \tag{4.5}$$

$$\|\tfrac{1}{2}\, dA\|_1 = 1. \tag{4.6}$$

Proof. The linear operator $\frac{1}{2}\, dA$ is represented by the bi-infinite matrix C_{ij} given by

$$C_{ij} = c_{i-2j}. \tag{4.7}$$

Thus from the positivity of $\{c_i\}_{i=0}^{k-1}$ we get

$$\|\tfrac{1}{2}\, dA\|_\infty = \sup_i \left(\sum_j |c_{i-2j}|\right) = \max\left(\sum c_{2i}, \sum c_{2i+1}\right) < 1$$

and

$$\|\tfrac{1}{2}\, dA\|_1 = \sup_j \left(\sum_i |c_{i-2j}|\right) = \sum_i c_i = 1. \quad \blacksquare$$

By applying Theorem 2.4 together with (4.5) we get:

THEOREM 4.2. *Let $A \in P^1$ then A is C^0 and the refinable function of A, $E(t)$ satisfies*

$$E \in \mathrm{LIP}_\gamma, \qquad \gamma = \log_2(1/\|\tfrac{1}{2}\, dA\|_\infty). \tag{4.8}$$

The estimation of γ in (4.8) may be refined by considering $(\frac{1}{2}\, dA)^L$, $L > 1$ (see Remark 2.5).

For the following monotonicity preserving theorems, we assume that $\{f_i^0\}_{i=1}^r \subset R$ is a non-decreasing sequence. The convergence interval of the limit function f is $[k, r+1]$, where we assume $r \geqslant k$.

The support of the piecewise linear interpolant f^n is given by $[k-(k-1)/2^n, r+1-(1/2^n)]$. In order to define f^n on the whole interval of convergence we add a dummy value $f_{r+1}^0 := f_r^0$ which does not affect the limit on $[k, r+1]$, and from now on we assume

$$[k, r+1] \subset \mathrm{supp}(f^n). \tag{4.9}$$

THEOREM 4.3. *Let $A \in P^1$. If $\{f_i^0\}_{i=1}^r$ is non-decreasing, then the limit function is non-decreasing on $[k, r+1]$.*

Proof. Since $\{\Delta f_i^0\}$ is a non-negative sequence and the transformation $\{\Delta f_i^n\} \to \{\Delta f_i^{n+1}\}$ is by a non-negative matrix then it is clear that $\{f^n\}$ is a sequence of non-decreasing functions on $[k, r+1]$ and the limit function is also non-decreasing. ∎

The next theorem provides strict monotonicity.

THEOREM 4.4. *Let* $A \in P^1$ *and let* $\{f_i^0\}_{i=1}^r$ *be non-decreasing. If* $f_i^0 < f_{i+k-1}^0$ *for each* i, $1 \leqslant i \leqslant r-k+1$, *then* f *is strictly increasing on* $[k, r+1]$.

Proof. First we prove that each $k-1$ successive differences at level n have a non-zero element (which is the case for $n=0$). Each $k-1$ differences at level n, $\{\Delta f_i^n\}_{i=j}^{j+k-2}$, produces k differences at level $n+1$, $\{\Delta f_i^{n+1}\}_{i=2j+k-2}^{2j+2k-3}$, according to transformation (4.1) given by the matrix:

$$\begin{pmatrix} c_{k-2} & c_{k-4} & \cdots & \cdots & \cdots & \cdots & \cdots \\ c_{k-1} & c_{k-3} & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & c_{k-2} & c_{k-4} & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & c_3 & c_1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & c_2 & c_0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & c_3 & c_1 \end{pmatrix}, \tag{4.10}$$

Assume by induction that $\{\Delta f_i^n\}_{i=j}^{j+k-2}$ has a positive element; then also $\{\Delta f_i^{n+1}\}_{i=2j+k-2}^{2j+2k-4}$ and $\{\Delta f_i^{n+1}\}_{i=2j+k-1}^{2j+2k-3}$ have a positive element, since each column of the matrix (4.10) has at least two positive entries. Now, assume in contradiction that there exist t_1 and t_2 such that $k \leqslant t_1 < t_2 \leqslant r+1$ and $f(t_1) = f(t_2)$. For n sufficiently large there exist $j_1, j_2 \in Z^+$ such that $j_1 + k - 1 < j_2$, $f(t_1) \in \mathrm{CONV}(\{f_i^n\})_{i=j_1}^{j_1+k-1}$ and $f(t_2) \in \mathrm{CONV}(\{f_i^n\})_{i=j_2}^{j_2+k-1}$. Since $f_{j_1+k-1}^n < f_{j_2}^n$ it follows that $f(t_1) < f(t_2)$. ∎

Remark 4.5. If $A \in P^m$ then Theorem 4.3 holds also if we substitute f by $f^{(m-1)}$ and $\{f_i^0\}$ by $\{\Delta^{m-1} f_i^0\}$. Theorem 4.4 for P^m schemes requires in addition $\Delta^{m-1} f_i^0 < \Delta^{m-1} f_{i+k-m}^0$, in order to produce $f^{(m-1)}$ which is strictly increasing. This extension is a corollary of Theorem 2.3 and (1.9). In particular P^2 schemes preserve the monotonicity and convexity of the initial values $\{f_i^0\}_{i=1}^r \subset R$.

THEOREM 4.6. *Let* $A \in P^m$, $\{f_i^0\}_{i=1}^r \subset R$ *then* $f^{(m-1)}$ *is of bounded variation and* $f^{(m)} \in L^1$.

Proof. By (1.9) and Theorem 2.3 it is sufficient to prove the case $m=1$. $\{f_i^0\}_{i=1}^r$ can be expressed as a linear combination of r monotone vectors, and it is clear that f is a linear combination of r monotone functions which implies $f \in BV$. If $f \in BV$ then it is well known that $f' \in L^1$. ∎

Remark 4.7 (Geometric Properties of P^1 Schemes). It is well known that a curve in R^d has a finite arc-length if and only if the component functions are BV. Thus, Theorem 4.6 states a geometric property of P^1 schemes. Theorems 4.3 and 4.4 also have a geometric interpretation. Assume that the control points at level n, $\{f_i^n\}$, lie on a straight line in R^d. For any C^0 scheme it is true that $\{f_i^{n+1}\}$, lie on the same line. P^1 schemes have an order preserving property, i.e., if $\{f_i^n\}$ appears in a natural order then $\{f_i^{n+1}\}$ appears also in a natural order.

In the following we analyse variation diminishing properties. Here $\{f_i^0\}_{i=1}^r \subset R^d$ and we define the "length" of a control polygon at level n by

$$L(f^n) = \sum_i \|\Delta f_i^n\|, \tag{4.11}$$

where $\|\ \|$ denotes any semi-norm in R^d.

THEOREM 4.8. *Let $A \in P^1$ and $\{f_i^0\}_{i=1}^r \subset R^d$; then*

$$L(f^{n+1}) \leqslant L(f^n).$$

Proof. In view of (4.1) together with the triangle inequality we get

$$\|\Delta f_i^{n+1}\| \leqslant \sum_{j \in Z} c_{i-2j} \|\Delta f_j^n\|. \tag{4.12}$$

The proof is completed by (4.6). ∎

Remark. Let $L^{(j)}(f^n)$ be defined by

$$L^{(j)}(f^n) = \sum_i 2^{jn} \|\Delta^{j+1} f_i^n\|; \tag{4.13}$$

then the extension of Theorem 4.8 for P^m schemes is

$$L^{(j)}(f^{n+1}) \leqslant L^{(j)}(f^n), \qquad j = 0, ..., m-1, \tag{4.14}$$

Theorem 4.8 is especially significant when applied to the norm $\|\ \|_1$ in R and to the norm $\|\ \|_2$ in R^d.

THEOREM 4.10. *Let $A \in P^1$, $\{f_i^0\} \subset R$ and let $T_k^{r+1}(f)$ denote the total variation of f on $[k, r+1]$. Then*

$$T_k^{r+1}(f) \leqslant \sum_{i=2}^{r} |\Delta f_i^0|. \tag{4.15}$$

Proof. Let $[a_n, b_n]$ denote the support of the piecewise linear function f^n. The total variation of f^n on this interval is given by

$$T_{a_n}^{b_n}(f^n) = \sum_i |\Delta f_i^n|. \tag{4.16}$$

In view of Theorem 4.8, we have

$$T_{a_{n+1}}^{b_{n+1}}(f^{n+1}) \leqslant T_{a_n}^{b_n}(f^n), \tag{4.17}$$

and since by (4.9), $[k, r+1] \subset [a_n, b_n]$, then it follows that

$$T_k^{r+1}(f^n) \leqslant \sum_{i=0}^{r-1} |\Delta f_i^0|. \tag{4.18}$$

Since $f(t) = \lim_{n \to \infty} f^n(t)$ on $[k, r+1]$ then by a Lemma on p. 100 of [11] we get

$$T_k^{r+1}(f) \leqslant \underline{\lim}\, T_k^{r+1}(f^n) \tag{4.19}$$

which completes the proof. ∎

By the same argument we get:

THEOREM 4.11. *Let $A \in P^1$, $\{f_i^0\}_{i=1}^r \subset R^d$ and let $L_k^{r+1}(f)$ denote the arc-length of the limit curve. Then*

$$L_k^{r+1}(f) \leqslant \sum_{i=2}^{r} \|\Delta f_i^0\|_2. \tag{4.20}$$

Remark 4.12. The analysis of this section up to now holds for a weaker definition of P^m. This definition requires that $d^{m-1}A \in C^0$ and the existence of some L such that the L-iterated scheme of $d^m A$ has a positive mask (see Remark 2.5). In fact, there are schemes where the mask of $d^m A$ includes negative numbers and these schemes are P^m corresponding to the new definition.

In the following, we discuss the zeros of the polynomial $A(z) = \sum_{j=0}^k a_j z^j$ of a P^m scheme. It is clear that $z = -1$ is an m-fold zero and we are interested in the zeros of

$$B(z) = 2^m (z+1)^{-m} A(z) = \sum_{j=0}^{k-m} b_j z^j, \qquad b_j > 0. \tag{4.21}$$

If the zeros of $B(z)$ satisfy $\mathrm{Re}(z) < 0$ then obviously $b_j > 0$, but the converse is not true. Let $b_0 = 1 - \varepsilon$, $b_{k-m} = 1 - \varepsilon$, $b_1 = \cdots = b_{k-m-1} =$

$2\varepsilon/(k-m-1)$, then for ε sufficiently small $B(z)$ has a zero sufficiently close to $e^{i\pi/(k-m)}$. The following theorem restricts the location of the zeros, and it is a direct application of Theorem 3.1, p. 397 in [8].

THEOREM 4.13. *Let $A(z)=\sum_{j=0}^{k} a_j z^j$ be a P^m scheme then $z=-1$ is an m-fold zero and the other zeros are in the sector $|\arg z| \geqslant \pi/(k-m)$.*

5. Bell-Shaped Refinable Functions

DEFINITION 5.1. A sequence $\{a_i\}_{i=0}^{k}$ ($a_i=0$, $i<0$, $i>k$) is bell-shaped if

$$a_i>0, \qquad 0\leqslant i\leqslant k; \tag{5.1}$$

$$a_i=a_{k-i}, \qquad 0\leqslant i\leqslant k; \tag{5.2}$$

$$a_0<a_1<\cdots<a_{[k/2]}. \tag{5.3}$$

For example, Chaikin's scheme has a bell-shaped mask.

THEOREM 5.1. *Let $\{a_i\}_{i=0}^{k}$ be bell-shaped and satisfy* (1.5); *then the scheme is P^1.*

Proof. Let $\{c_i\}_{i=0}^{k-1}$ be the mask of the difference process. By (4.2) we get

$$\begin{aligned} c_0&=a_0\\ c_1&=a_1-a_0\\ c_2&=a_2-a_1+a_0\\ &\vdots\\ c_{[k/2]}&=a_{[k/2]}-a_{[k/2]-1}+\cdots+(-1)^{[k/2]}a_0 \end{aligned} \tag{5.4}$$

and $c_0, ..., c_{[k/2]}$ are positive. Analogously, $c_{[k/2]+1}, ..., c_k$ are positive. ∎

THEOREM 5.2. *Let $\{c_i\}_{i=0}^{k-1}$ be bell-shaped; then the mask $\{a_i\}_{i=0}^{k}$ is bell-shaped.*

Proof. The mask $\{a_i\}_{i=0}^{k}$ is defined by

$$a_i=c_i+c_{i-1}, \tag{5.5}$$

thus (5.1) and (5.2) are obvious. Equation (5.3) is an immediate result of the following relations:

$$\begin{gathered} c_{i-1} < c_i \leqslant c_{i+1} \Rightarrow a_i < a_{i+1} \\ c_{i-1} < c_i > c_{i+1}, \qquad c_{i-1} = c_{i+1} \Rightarrow a_i = a_{i+1}. \end{gathered} \tag{5.6}$$ ∎

THEOREM 5.3. *Let* $\{a_i\}_{i=0}^{k}$ *be bell-shaped. Define* $g(t)$ *by*

$$g(t) = E(t) - E(t-1), \tag{5.7}$$

where $E(t)$ *is the associated refinable function. Then*

$$\begin{aligned} g(t) &> 0 \qquad 0 < t < \frac{k+1}{2} \\ g(t) &< 0 \qquad \frac{k+1}{2} < t < k+1 \\ g(t) &= 0 \qquad \textit{otherwise}. \end{aligned} \tag{5.8}$$

Before proceeding with the details of the proof, we comment on symmetric properties. The fact that a C^0 scheme with a symmetric mask $(a_i = a_{k-i})$ produces a symmetric refinable function $(E(k/2) - t) = E((k/2) + t))$ is obvious. The function $g(t)$ is anti-symmetric about $(k+1)/2$, since by Theorem 2.3

$$g(t) = F'(t), \tag{5.9}$$

where $F(t)$ is the refinable function of the scheme with the mask $\tilde{a}_i = \frac{1}{2}(a_{i-1} + a_i)$, $i = 0, ..., k+1$, and by Theorem 5.2 $\{\tilde{a}_i\}_{i=0}^{k+1}$ is symmetric.

In the following we prove Theorem 5.3 for an odd number k (the even case is proved by analogous arguments). For the proof we need the following lemma.

LEMMA 5.4. *Let* $\{a_i\}_{i=0}^{k}$ *be a bell-shaped sequence, and assume that* k *is odd. Let* B *denote the matrix*

$$b_{i,j} = a_{i-2j+k}, \qquad 1 \leqslant i, \quad j \leqslant k-1, \tag{5.10}$$

and let the vector $X = (x_1, ..., x_{k-1})$ *satisfy*

$$x_i = -x_{k-i}, \tag{5.11}$$

and

$$x_i > 0, \qquad 1 \leqslant i \leqslant \frac{k-1}{2}. \tag{5.12}$$

Then the vector $Y=(y_1, \ldots, y_{k-1})$ *given by*

$$Y=BX, \tag{5.13}$$

satisfies

$$y_i=-y_{k-i} \tag{5.14}$$

$$y_i>0, \qquad 1\leqslant i\leqslant\frac{k-1}{2}. \tag{5.15}$$

Proof. For convenience we first illustrated the case $k=5$. The matrix B is given by

$$B=\begin{pmatrix} a_4 & a_2 & a_0 & 0 \\ a_5 & a_3 & a_1 & 0 \\ 0 & a_4 & a_2 & a_0 \\ 0 & a_5 & a_3 & a_1 \end{pmatrix}.$$

The vector $X=(x_1, x_2, x_3, x_4)$ satisfies

$$0<x_1, \qquad 0<x_2, \qquad x_3=-x_2, \qquad x_4=-x_3,$$

and we state that $Y=BX$ satisfies

$$0<y_1, \qquad 0<y_2, \qquad y_3=-y_2, \qquad y_4=-y_1.$$

Note that B is obtained by deleting the last column and the two last rows of the matrix A in (2.1).

A main observation is that

$$b_{i,j}=b_{k-i,k-j}, \tag{5.16}$$

which follows from the symmetry of the mask, i.e., $a_i=a_{k-i}$. Now, since

$$y_i=\sum_{j=1}^{k-1} b_{ij}x_j, \tag{5.17}$$

then in view of (5.11) and (5.16) we obtain

$$\begin{aligned} y_{k-i} &= \sum_{j=1}^{k-1} b_{k-i,j}x_j=\sum_{j=1}^{k-1} b_{i,k-j}x_j=-\sum_{j=1}^{k-1} b_{i,k-j}x_{k-j} \\ &= -\sum_{j=1}^{k-1} b_{i,j}x_j=-y_i, \end{aligned} \tag{5.18}$$

which proves (5.14).

Another result of (5.11) and (5.16) is

$$y_i = \sum_{j=1}^{(k-1)/2} (b_{ij} - b_{i,k-j})\, x_j, \qquad 1 \leqslant i \leqslant k-1, \tag{5.19}$$

and note that in the above sum $x_j > 0$ by (5.12). In order to prove (5.15) we will use also property (5.3) of $\{a_i\}$. In particular (5.3) yields

$$a_i > a_j \Leftrightarrow \left| i - \frac{k}{2} \right| < \left| j - \frac{k}{2} \right|. \tag{5.20}$$

Now since

$$\begin{aligned} b_{i,j} &= a_{i+(k-2j)} \\ b_{i,k-j} &= a_{i-(k-2j)}, \end{aligned} \tag{5.21}$$

then for $0 \leqslant i, j \leqslant (k-1)/2$ we get $k - 2j > 0$ and as a result of (5.20), $b_{i,j} > b_{i,k-j}$, which implies the positivity of (5.19) for $1 \leqslant i \leqslant (k-1)/2$. ∎

Proof of Theorem 5.3. In the following we assume that k is odd. The function $g(t)$ on its support $(0, k+1)$ is the limit corresponding to the initial data

$$\underbrace{0, \ldots, 0}_{k-1}, 1, -1, \underbrace{0, \ldots, 0}_{k-1}. \tag{5.22}$$

Applying the scheme once we obtain $\{f_i^1\}$:

$$\underbrace{0, \ldots, 0}_{k-1}, a_0, a_1, \ldots, a_i - a_{i-2}, \ldots, -a_{k-1}, -a_k, \underbrace{0, \ldots, 0}_{k-1}. \tag{5.23}$$

We now split (5.23) into six regions:

$$\underbrace{0}_{Q_1} \;\; \underbrace{+}_{Q_2} \;\; \underbrace{+}_{Q_3} \;\; \vdots \;\; \underbrace{-}_{Q_4} \;\; \underbrace{-}_{Q_5} \;\; \underbrace{0}_{Q_6}. \tag{5.24}$$

Q_1 and Q_6 contain $k-1$ zeros each, Q_3 (Q_4) contains $(k-1)/2$ positive (negative) terms, and Q_2 (Q_5) contains two positive (negative) terms. The positivity (negativity) of Q_3 (Q_4) is established by (5.20). Observe that the sequence (5.24) is anti-symmetric about the dashed line separating between Q_3 and Q_4, a fact which follows directly by (5.2).

Assume by induction that the data $\{f_i^n\}$ at level n has the following structure. Q_1^n (Q_6^n) is a boundary layer of $k-1$ zeros. The other values are split as in (5.24) into two consecutive sets of positive and negative numbers. The last $(k-1)/2$ positive numbers are denoted by Q_3^n and the others (which include at least two numbers) are Q_2^n. Q_4^n, Q_5^n are defined analogously. Also we assume that Q_3^n and Q_4^n are anti-symmetric about the dashed line separating them.

Now, apply one step of the scheme and observe that each value at level $n+1$ is a combination of $(k+1)/2$ values at level n. Hence values from Q_2^n and Q_4^n (Q_3^n and Q_5^n) do not appear in the same combination and by Lemma 5.4, $Q_3^n \cup Q_4^n$ is reproducing its structure. Combinations generated by $Q_2^n \cup Q_3^n$ are clearly positive and common combinations of Q_1^n and $Q_2^n \cup Q_3^n$ are also positive. Since there exist only $k-1$ combinations of Q_1^n alone, then Q_2^{n+1} at level $n+1$ has only positive values and the induction is completed. $Q_1^n \cup Q_2^n \cup Q_3^n$ at level n determine $g(t)$ on $(0, (k+1)/2 - \varepsilon_n)$, where $\varepsilon_n \to 0$ as $n \to \infty$. From the positivity of Q_2^n it follows that $g > 0$ on $(0, (k+1)/2)$.

THEOREM 5.5. *Let dA have a bell-shaped mask satisfying* (1.5), *then the refinable function of A, $E(t)$ is bell-shaped* (*according to Definition* 1.3).

Proof. By Theorem 5.1 dA is P^1, hence A is P^2 and $E(t) \in C^1$. $E(t) > 0$ on $(0, k)$ by Theorem 3.1 and is symmetric about $k/2$ since the mask of A is symmetric. Let $\tilde{E}(t)$ be the refinable function of dA then by Theorem 2.3

$$E'(A) = \tilde{E}(t) - \tilde{E}(t-1), \tag{5.25}$$

and by Theorem 5.3 the proof is completed. ∎

Remark 5.6. Our interest in bell-shaped functions follows from the following reason. Assume that $E(t)$ is bell-shaped; then the curve

$$\sum_{i=1}^{r} f_i^0 E(t-i), \qquad t \in [k, r+1], \tag{5.26}$$

or the tensor-product surface

$$\sum_{i=1}^{r_1} \sum_{j=1}^{r_2} f_{ij}^0 E(u-i)\, E(v-j), \qquad u \in [k, r_1+1), \quad v \in [k, r_2+1], \tag{5.27}$$

have no self-intersections and critical points if certain geometrical conditions are imposed on $\{f_i^0\}_{i=1}^r$, $\{f_{ij}^0\}_{i=1}^{r_1}{}_{j=1}^{r_2}$, respectively (see [7]). In particular, B-splines are bell-shaped.

ACKNOWLEDGMENTS

I thank N. Dyn and D. Levin for useful discussions and for their helpful comments.

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